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# Black holes and black strings of $N = 2$, $d = 5$ supergravity in the H-FGK formalism

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**Abstract**

We study general classes and properties of extremal and non-extremal static black-hole solutions of $N = 2$, $d = 5$ supergravity coupled to vector multiplets using the recently proposed H-FGK formalism, which we also extend to static black strings. We explain how to determine in general the integration constants and physical parameters of the black-hole and black-string solutions. We derive some model-independent statements, including the transformation of non-extremal flow equations to the form of those for the extremal flow. We apply our methods to the construction of example solutions (among others a new extremal string solution of heterotic string theory on $K_3 \times S^1$) and analyze their properties. In all the cases studied the product of areas of the inner and outer horizon of a non-extremal solution coincides with the square of the moduli-independent area of the horizon of the extremal solution with the same charges.

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# 1 Introduction

Static, spherically symmetric black-hole solutions of $N = 2$ supergravity can be conveniently studied in the effective black-hole potential formalism originally developed by Ferrara, Gibbons and Kallosh [1] in four dimensions, later extended to arbitrary dimensions [2], as well as to $p$-branes [3]. This is especially true for supersymmetric extremal solutions, where expressing the effective potential by the central charge leads to the derivation of first-order flow equations for the scalars (implied by the Killing spinor equations), whose attractor fixed points (corresponding to the sets of values of scalars on the event horizon) [4] are determined by critical points of the central charge.

Even though not all extremal black holes are supersymmetric [5, 6] and all non-extremal black holes break all supersymmetries, it turns out [7] that analogous flow equations may be derived for four- [8] and five-dimensional extremal black holes [9], as well as for non-extremal $p$-branes [10] and black holes [11]. This fact alone already hints at a possibility that perhaps non-supersymmetric solutions could be obtained in a similar way to the supersymmetric ones. Indeed, at least in a class of four- [12] and five-dimensional [13, 2] black holes and $p$-branes [3], the known supersymmetric solution can be deformed to a unique non-extremal solution, from which both supersymmetric and non-supersymmetric extremal solutions are recovered in the different limits in which the non-extremality

parameter vanishes. This is, as far as we know, the only systematic method for constructing general extremal non-supersymmetric black-hole solutions, in particular when the black-hole potential has flat directions and the values of the scalar fields on the horizon have some dependence on the asymptotic values [12].

That a deformation from a supersymmetric to a non-extremal solution must be possible and that all static solutions with spherical symmetry can be treated in the same manner, becomes clear in a new set of $H$-variables introduced in the 5-dimensional case in [14, 13][5] and in the 4-dimensional case in [16, 15][6] in which all static, spherically symmetric black-hole solutions of a given model take the same, universal functional form, irrespective of supersymmetry or extremality, although the radial profile of the $H$-variables themselves will be different for the different kinds of solutions. These variables arise naturally in the classification of the timelike supersymmetric solutions of these theories [17, 18, 19], to which the supersymmetric black holes belong, but also occur in the classification of the timelike supersymmetric solutions of more general theories (with hypermultiplets [20, 21], gaugings [22] or both [23, 24] or with both and additionally tensor multiplets [25]) and transform linearly under the duality transformations (subgroups of $Sp(2n + 2, \mathbb{R})$ in $d = 4$ and $SO(n + 1)$ in $d = 5$, for $n$ vector multiplets). These variables replace the scalars and the metric function of the theory that appear on different footing in the effective action and their use should, in principle, simplify and systematize the task of constructing explicit solutions and general results.

In the present work we use the 5-dimensional version of this formalism to ask general questions about the black-hole solutions of $N = 2$, $d = 5$ supergravity and to construct some families of solutions. Furthermore, profiting from the recent extension of the FGK formalism to $p$-branes in any dimension, we extend the H-FGK formalism to cover the case of black strings in these theories, introducing new $H$-variables inspired by the classification of the null supersymmetric solutions of $N = 2$, $d = 5$ supergravities [26, 17, 18, 21, 25]. We then study the resulting system as we do with the one for black holes.

We start by reviewing the H-FGK formalism for black holes of $N = 2$, $d = 5$ supergravity coupled to vector multiplets in section 2.1. Following [15], we introduce the basic definitions concerning the theories we deal with, the metric ansatz and the $H$-variables. We show how we can get the metric that covers the region lying between the inner (Cauchy) horizon and the singularity (not discussed in [2, 3]) from the one that covers the exterior of the outer (event) horizon in the present 5-dimensional case. This will allow us to compute the "entropy" and "temperature" associated with the inner horizon[7].

In section 2.2 we will apply the formalism just discussed to the study of extremal (BPS and non-BPS) black holes under the reasonable assumption that, for extremal black holes, all the $H$-variables are harmonic functions in the transverse space, defined by two integration constants. We recover the results obtained in [13] and find some new ones. We study how these integration constants can be determined as functions of the physical parameters in general, finding that half of them are always determined by the asymptotic values of the scalars, that can be fixed at will. The other half play the rôle of "fake charges" and many physical quantities (mass, entropy) are determined by the *fake central charge* (or superpotential) constructed by the standard formula with the charges replaced by the fake charges. In the extremal case the fake charges can be determined by extremization of the black-hole potential on the horizon, like in the original FGK formulation, but with the actual black-hole potential now understood as a function of the fake and physical charges, rather than of scalars and physical charges. The first-order flow equations for extremal black holes are constructed in section 2.3 using

[5] A different derivation specific for $N = 2$ $d = 5$ supergravity theories was also given in [15].

[6] Again, the derivation of [15] makes heavy use of the formalism of $N = 2$, $D = 4$ supergravity.

[7] The inner horizon is also reached, albeit in a different way, in ref. [13].

the simple procedure proposed in [27], which is valid for non-supersymmetric cases as well. The equations of motion of an extremal black hole are reproduced when the *fake black-hole potential* (a function of scalars and fake charges) is equal to the true one.

We then go on to study the non-extremal case in section 2.4, adopting for the $H$-variables the exponential or hyperbolic ansatz of [13, 12, 2]. We show how the relation between extremization of the black-hole potential and attractor behavior for the scalars and the relation between entropy and black-hole potential on the horizon are modified in the non-extremal case. In section 2.5 we demonstrate how the first-order equations for non-extremal black holes can be brought to the form of the extremal flow.

Explicit solutions are given in section 3. The examples that we analyze include the general non-extremal black holes with constant scalars, in section 3.1 (found earlier in [13] in a different way), the $STU$ model, in section 3.2, which we solve paying particular attention to the possible signs of the charges, and in section 3.3 the models of the reducible Jordan sequence, whose black-hole potential has flat directions and whose values of scalars on the horizon, in some non-supersymmetric cases, are not completely fixed by the charges.

Finally, in section 4 we generalize this approach to black strings, using the extension, recently constructed in [3], of the FGK formalism to $p$-branes, and introducing dual $H$-variables (which we shall denote by $K$). Our study of this case follows what we did for the black holes in the previous sections: we find the general solutions for non-extremal black strings with constant scalars for any $N = 2$, $d = 5$ supergravity theory in section 4.2, derive flow equations for black strings in section 4.1, and construct explicitly the extremal black strings of the pure and the heterotic $STU$ model in section 4.3.

Section 5 contains our conclusions.

## 2 The H-FGK formalism in five dimensions

### 2.1 H-variables

We start by recalling the salient points of ref. [15]. $N = 2$, $d = 5$ supergravity [28] coupled to $n$ vector multiplets contains, apart from the metric, $n$ scalar fields $\phi^x$ ($x = 1, \ldots, n$) and $n + 1$ vector fields $A^I$ ($I = 0, \ldots n$). The coupling between these fields is specified by real special geometry, which in itself can be formulated in terms of a constant completely symmetric real tensor $C_{IJK}$ and a section $h^I(\phi)$ that obeys the fundamental constraint

$$\mathcal{V}(h) = C_{IJK}h^I h^J h^K = 1\,. \tag{2.1}$$

If we then define the derived objects

$$h_I \equiv C_{IJK}h^J h^K\,, \qquad h^I_x \equiv -\sqrt{3}\,\frac{\partial h^I}{\partial \phi^x} \qquad \text{and} \qquad h_{Ix} \equiv \sqrt{3}\,\frac{\partial h_I}{\partial \phi^x}\,, \tag{2.2}$$

we can see that they satisfy the following relations

$$h^I h_I = 1 \qquad \text{and} \qquad h^I h_{Ix} = h_I h^I_x = 0\,. \tag{2.3}$$

The metric on the scalar manifold, $g_{xy}$, and the vector kinetic matrix, $a_{IJ}$, are given by

$$g_{xy} = h_{Ix}h^I_y \qquad \text{and} \qquad a_{IJ} = 3h_I h_J - 2C_{IJK}h^K = h_I h_J + h_{Ix}h^x_J\,. \tag{2.4}$$

With these definitions we can write the bosonic part of the action for $N = 2$, $d = 5$ supergravity coupled to $n$ vector supermultiplets as

$$\mathcal{I}_5 = \int_5 \left( R \star 1 + \tfrac{1}{2} g_{xy}\, d\phi^x \wedge \star d\phi^y - \tfrac{1}{2} a_{IJ} F^I \wedge \star F^J + \tfrac{1}{3\sqrt{3}} C_{IJK} F^I \wedge F^J \wedge A^K \right) . \tag{2.5}$$

Having briefly detailed the relevant physical theory that we want to consider, we can discuss the FGK formalism.

The starting point of the FGK formalism in 5 dimensions is the ansatz for a spherically symmetric metric describing the exterior of the event horizon of a generic 5-dimensional black hole, namely

$$ds^2 = e^{2U(\rho)} dt^2 - e^{-U(\rho)} \left( \frac{\mathcal{B}^3}{4 \sinh^3(\mathcal{B}\rho)}\, d\rho^2 + \frac{\mathcal{B}}{\sinh(\mathcal{B}\rho)}\, d\Omega^2_{(3)} \right) , \tag{2.6}$$

where $d\Omega^2_{(3)}$ is the round metric on the 3-sphere of unit radius and $\mathcal{B}$ is the so-called *non-extremality parameter*, meaning that extremal solutions are obtained as the $\mathcal{B} \to 0$ limit. In the employed coordinate system the asymptotic region lies at $\rho = 0$, whereas the putative horizon is located at $\rho \to \infty$: in order for the metric (2.6) to describe a non-extremal black hole, the function $U$ must have the following limiting behavior

$$\lim_{\rho \to \infty} e^{-U} = e^{\mathcal{B}\rho} , \tag{2.7}$$

which ensures that the limiting spacetime is a 2-dimensional Rindler space times a 3-sphere.

Although this was not realized in [2], the same general metric describes the interior of the inner (Cauchy) horizon as well, just as it happens in $d = 4$ dimensions [12], although in this case it is more difficult to see. Given a regular solution of the above form describing the exterior of the black hole for $\rho \in (0, +\infty)$, we can obtain the metric that describes the interior of the inner horizon by transforming[8] that metric according to

$$\rho \longrightarrow -\varrho , \qquad e^{-U(\rho)} \longrightarrow -e^{-U(-\varrho)} . \tag{2.8}$$

The new metric has the same general form in terms of the coordinate which now takes values in the range $\varrho \in (\varrho_{\text{sing}}, +\infty)$ because the metric will generically hit a singularity before $\varrho$ reaches $0$: if the original $e^{-U}$ is always finite for positive values of $\rho$, the transformed one will have a zero for some finite positive value of $\varrho$, as we will see in the examples.

Being interested in spherically symmetric solutions black hole solutions we take $\phi^x = \phi^x(\rho)$ and can solve the vector field equations of motion by putting

$$F^I = -\sqrt{3}\, e^{2U} a^{IJ} q_J\, dt \wedge d\rho , \tag{2.9}$$

where the $q$'s are the electric charges. Using the ansätze (2.6) and (2.9) in the remaining equations of motion, we see that they all reduce to the following equations

$$\ddot{U} + e^{2U} V_{\text{bh}}(\phi, q) = 0 , \tag{2.10}$$

$$\ddot{\phi}^x + \Gamma_{yz}{}^x \dot{\phi}^y \dot{\phi}^z + \tfrac{3}{2}\, e^{2U} \partial^x V_{\text{bh}}(\phi, q) = 0 , \tag{2.11}$$

$$\dot{U}^2 + \tfrac{1}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y + e^{2U} V_{\text{bh}}(\phi, q) - \mathcal{B}^2 = 0 , \tag{2.12}$$

[8] This is *not* a coordinate transformation, because, among other reasons, it relates the metric in two different, disjoint patches of the spacetime.

where we used the over-dot to denote derivation with respect to $\rho$ and we defined the *black hole potential* by

$$V_{\rm bh}(\phi,q) \equiv -a^{IJ}q_Iq_J = -\mathcal{Z}_{\rm e}^2 - 3\,\partial_x\mathcal{Z}_{\rm e}\,\partial^x\mathcal{Z}_{\rm e}\,, \tag{2.13}$$

and in the last step defined the *(electric) central charge* by $\mathcal{Z}_{\rm e} = \mathcal{Z}_{\rm e}(\phi,q) \equiv h^I q_I$.

The equations (2.10) and (2.11) can be obtained from the FGK effective action

$$\mathcal{I}[U,\phi^x] = \int d\rho\left(\dot{U}^2 + a^{IJ}\dot{h}_I\dot{h}_J - e^{2U}V_{\rm bh}(\phi,q) + \mathcal{B}^2\right), \tag{2.14}$$

where we made use of eqs. (2.2) and (2.4) to cast it into a more suitable form. Given this action, eq. (2.12) can be interpreted as the constraint that the Hamiltonian corresponding to eq. (2.14) be zero.

At this point we introduce two new sets of variables, $\tilde{H}^I$ and $H_I$, related to the original ones $(U,\phi^x)$ by

$$e^{-U/2}h^I(\phi) \equiv \tilde{H}^I\,, \tag{2.15}$$

$$e^{-U}h_I(\phi) \equiv H_I\,, \tag{2.16}$$

and two new functions $\mathsf{V}$ and $\mathsf{W}$

$$\mathsf{V}(\tilde{H}) \equiv C_{IJK}\tilde{H}^I\tilde{H}^J\tilde{H}^K\,, \qquad \mathsf{W}(\tilde{H}) = 2\,\mathsf{V}(\tilde{H})\,, \tag{2.17}$$

but which are not constrained. Using the homogeneity properties of these functions we find that

$$e^{-\frac{3}{2}U} = \tfrac{1}{2}\mathsf{W}(H)\,, \tag{2.18}$$

$$h_I = (\mathsf{W}/2)^{-2/3}H_I\,, \tag{2.19}$$

$$h^I = (\mathsf{W}/2)^{-1/3}\tilde{H}^I\,. \tag{2.20}$$

We can use these formulae to perform the change of variables in the effective action for static, spherically symmetric black holes of $N=2, d=5$ supergravity [2], which can be rewritten in the convenient form

$$\mathcal{I}[U,\phi^x] = \int d\rho\left(\dot{U}^2 + a^{IJ}\dot{h}_I\dot{h}_J + e^{2U}a^{IJ}q_Iq_J + \mathcal{B}^2\right). \tag{2.21}$$

Thanks to the identity

$$a^{IJ} = -\frac{3}{2}\left(\frac{\mathsf{W}}{2}\right)^{4/3}\partial^I\partial^J\log\mathsf{W} \tag{2.22}$$

the above action, in terms of the $H_I$ variables, becomes

$$-\tfrac{3}{2}\mathcal{I}[H] = \int d\rho\left(\partial^I\partial^J\log\mathsf{W}\left(\dot{H}_I\dot{H}_J + q_Iq_J\right) - \tfrac{3}{2}\mathcal{B}^2\right). \tag{2.23}$$

The equations of motion derived from the effective action are

$$\partial^K\partial^I\partial^J\log\mathsf{W}\left(H_I\ddot{H}_J - \dot{H}_I\dot{H}_J + q_Iq_J\right) = 0\,. \tag{2.24}$$

Multiplying these equations by $\dot{H}_K$ we get $\dot{\mathcal{H}} = 0$, the Hamiltonian constraint

$$\mathcal{H} \equiv \partial^I\partial^J\log\mathsf{W}\left(\dot{H}_I\dot{H}_J - q_Iq_J\right) + \tfrac{3}{2}\mathcal{B}^2 = 0\,, \tag{2.25}$$

where the integration constant has been set to $\frac{3}{2}\mathcal{B}^2$. Multiplying the equations of motion by $H_K$ we obtain

$$\partial^I\log\mathsf{W}\,\ddot{H}_I = \tfrac{3}{2}\mathcal{B}^2\,, \tag{2.26}$$

which is the equation of $U$ expressed in the new variables.

In the next subsections we shall use this formalism to study general families of solutions.

## 2.2 Extremal black holes

In the extremal black-hole case, $\mathcal{B} = 0$, we expect the $H_I$ to be harmonic functions on the transverse $\mathbb{R}^4$ space, *i.e.* linear functions of $\rho$

$$H_I = A_I + B_I \rho \,, \tag{2.27}$$

where the integration constants $A_I$ and $B_I$ are functions of the physical parameters (electric charges $q_I$ and asymptotic values of the scalars $\phi^x_\infty$) to be determined by requiring that the equations of motion are satisfied everywhere, *i.e.*

$$\partial^K \partial^I \partial^J \log \mathsf{W}(H)\,(B_I B_J - q_I q_J) = 0\,, \tag{2.28}$$

$$\partial^I \partial^J \log \mathsf{W}(H)\,(B_I B_J - q_I q_J) = 0\,, \tag{2.29}$$

and the physical fields are correctly normalized at spatial infinity ($\rho \to 0$). We study these conditions first.

Asymptotic flatness requires that $\mathsf{W}(H(0)) = 2$, *i.e.*[9]

$$\mathsf{W}(A) = 2\,. \tag{2.30}$$

In order to study the asymptotic behavior of the scalars it is convenient to introduce the following model-independent definition for the $n$ physical scalars of a generic $N = 2, d = 5$ theory:[10]

$$\varphi_x \equiv \frac{h_x}{h_0}\,. \tag{2.31}$$

The possible values of these scalars have to be determined model by model. We will ignore in this general discussion all issues related to their possible signs, singular values etc. Other choices amount to field redefinitions but eq. (2.31) allows us to write the solutions for the physical scalars in terms of the functions $H_I$ in a generic way as

$$\varphi_x = \frac{H_x}{H_0}\,. \tag{2.32}$$

Hence, the asymptotic values of all $H_I$, $I \neq 0$, are given by

$$A_x = \varphi_{x\,\infty} A_0\,. \tag{2.33}$$

Now, defining for convenience $\varphi_0 \equiv 1$, we can write

$$\mathsf{W}(\varphi) = \mathsf{W}(H/H_0) = H_0^{-3/2}\mathsf{W}(H)\,, \tag{2.34}$$

and taking into account the normalization at spatial infinity, we find that

$$A_0 = [\mathsf{W}(\varphi_\infty)/2]^{-2/3}\,. \tag{2.35}$$

Summarizing, we have shown that, in any model, the constants $A_I$ are given in terms of the asymptotic values of the scalars by

$$A_I = \varphi_{I\,\infty}[\mathsf{W}(\varphi_\infty)/2]^{-2/3}\,, \tag{2.36}$$

$$\varphi_{I\,\infty} = \frac{A_I}{A_0}\,. \tag{2.37}$$

[9] $\mathsf{W}(A)$ and other analogous expressions are to be understood as the functions one obtains when replacing the $H_I$ by the constants $A_I$.

[10] We use the symbol $\varphi$ to distinguish scalars defined by eq. (2.31) from physical scalars in an arbitrary, possibly different parametrization, which we denote by $\phi$. For an explicit example see the paragraph after eq. (3.28).

Finally, the mass, which in the extremal case is not an independent parameter, is given by $M = -\dot{U}(0)$. In terms of the integration constants:

$$M = \tilde{H}^I(A)B_I\,. \tag{2.38}$$

This expression can be rewritten as

$$M = h^I_\infty B_I \equiv \mathcal{Z}_{\rm e}(\varphi_\infty, B)\,, \tag{2.39}$$

by analogy with $\mathcal{Z}_{\rm e}(\varphi_\infty, q)$, the central charge of the theory. In general, the constants $B_I$ will not be equal to the electric charges $q_I$ and the *fake central charge* $\mathcal{Z}_{\rm e}(\varphi_\infty, B)$ will differ from the genuine supesymmetric central charge.

It is also possible and useful to derive generic expressions for the values of the scalars on the horizon ($\rho \to \infty$) and for the the hyperarea $\mathcal{A}$ of the horizon (or the Bekenstein-Hawking entropy $S = \mathcal{A}/4$) using the homogeneity properties of $\mathsf{W}$:

$$\varphi_{I\,\rm h} = \frac{B_I}{B_0}\,, \tag{2.40}$$

$$\frac{\mathcal{A}}{2\pi^2} = \frac{\mathsf{W}(B)}{2}\,. \tag{2.41}$$

Let us now study the equations of motion. First of all, notice that $B_I = \pm q_I$ always solves all the equations. These solutions may not be physically acceptable for certain sign choices, depending on the range of values that the scalars can take and the particular model.

The equation for $U$ (2.26) is automatically satisfied when the $H_I$ are harmonic. The near-horizon limit of the Hamiltonian constraint (2.25) reads

$$\partial^I\partial^J \log \mathsf{W}(B) q_I q_J \;=\; \partial^I\partial^J \log \mathsf{W}(B) B_I B_J \;=\; -3/2\,, \tag{2.42}$$

where the last step follows from homogeneity. The first term is equal to $\frac{3}{2}[\mathsf{W}(B)]^{-4/3} V_{\rm bh}(H,q)\big|_{\rm h}$, where $V_{\rm bh}(H,q)\big|_{\rm h} = V_{\rm bh}(B,q)$ is the value of the black-hole potential as a function of the $H_I$, evaluated on the horizon. Then, using eq. (2.41) we find that the entropy is given by the value of the black-hole potential on the horizon [1, 2]

$$\frac{\mathcal{A}}{2\pi^2} = [-V_{\rm bh}(B,q)]^{3/4}\,. \tag{2.43}$$

Similarly, in the near-horizon limit of the equations of motion we find that[11]

$$\partial^K V_{\rm bh}(B,q) = 0\,, \tag{2.44}$$

so the coefficients of $\rho$ in the harmonic functions are those that extremize the black-hole potential as a function of the $H_I$. From this result we can recover the known fact that the values of the scalars on the horizon are those that extremize the black-hole potential as a function of the scalars, using the fact that the black-hole potential is homogeneous of degree zero in the $H_I$.

The asymptotic limit ($\rho \to 0$) of the Hamiltonian constraint gives

$$\tfrac{3}{2}M^2 - \tfrac{1}{3}\partial^I\partial^J \log \mathsf{W}(A) B_I B_J + V_{\rm bh}(A,q) = 0\,. \tag{2.45}$$

[11] As a function of the $H$-variables the black-hole potential is scale-invariant and, therefore this equation determines the integration constants $B_I$ up to a common normalization factor which is, on the other hand, irrelevant to determine the entropy or the values of the scalars on the horizon. The normalization if fixed by the condition eq. (2.42).

Comparing this expression with the general Bogomol'nyi bound [1, 2]

$$M^2 + \tfrac{2}{3}g_{xy}(\phi_\infty)\Sigma^x\Sigma^y + V_{\rm bh}(\phi_\infty, q) = 0\,, \tag{2.46}$$

we get

$$\tfrac{1}{2}M^2 - \tfrac{1}{3}\partial^I\partial^J \log \mathsf{W}(A)B_I B_J = \tfrac{2}{3}g_{xy}(\varphi_\infty)\Sigma^x\Sigma^y\,. \tag{2.47}$$

These last two equations could be useful insofar as the the scalar charges $\Sigma^i = \Sigma^i(\varphi_\infty, q)$ were known, which is never the case until the full solution is known.

## 2.3 First-order flow equations for extremal black holes

Following ref. [27] it is easy to derive first-order equations for extremal (supersymmetric and non-supersymmetric) black holes for which the $H_I(\rho)$ are harmonic functions of the form given in eq. (2.27), where the integration constants $B_I$ extremize the black-hole potential as a function of the $H_I$, according to eq. (2.44).[12] We must insist in the fact that, due to the scale invariance of $V_{\rm bh}(H)$, the extremal values $B_I$ are defined only up to an overall multiplicative constant; this constant is, however, fixed by eq. (2.42)

We want to derive differential flow equations for the metric function $U$ and for the scalars $\phi^x$ (not necessarily parametrized as in eq. (2.31)). By virtue of eq. (2.3) and the definition of the variables $H_I$ we can write

$$de^{-U} = d(h^I h_I e^{-U}) = dh^I h_I e^{-U} + h^I d(h_I e^{-U}) = h^I d(h_I e^{-U}) = h^I dH_I\,, \tag{2.48}$$

Using then the harmonicity of the $H$'s, eq. (2.27), we arrive at

$$\frac{de^{-U}}{d\rho} = \mathcal{Z}_{\rm e}(\phi, B)\,. \tag{2.49}$$

Note that the above equation is given in terms of $\mathcal{Z}_{\rm e}(\phi, B)$ and not in terms of the theory's supersymmetric central charge $\mathcal{Z}_{\rm e}(\phi, q)$.

Similarly we can write

$$\begin{aligned} d\phi^x &= h^{Ix}h_{Iy}d\phi^y = -\sqrt{3}h^{Ix}\partial_y h_I d\phi^y = -\sqrt{3}h^{Ix}dh_I \\ &= -\sqrt{3}h^{Ix}d(e^U e^{-U}h_I) = -\sqrt{3}e^U h^{Ix}dH_I\,, \end{aligned} \tag{2.50}$$

which after a renewed call to the harmonicity of the $H$'s gives

$$\frac{d\phi^x}{d\rho} = -3e^U\partial^x\mathcal{Z}_{\rm e}(\phi, B)\,. \tag{2.51}$$

The fixed point of the flow, $\phi_{\rm h}(B)$, determined by the *fake charges* $B_I$ through the extremization of the black hole potential, eq. (2.44), is commonly called an attractor.

The flow equations (2.49) and (2.51) imply second-order equations which are identical to the equations of motion of the 5-dimensional effective FGK action [2], *i.e.* eqs. (2.10, 2.11), but with the scalar charges replaced in the black-hole potential by the constants $B_I$:

$$\ddot{U} + e^{2U}V_{\rm bh}(\phi, B) = 0\,, \tag{2.52}$$

$$\ddot{\phi}^x + \Gamma_{yz}{}^x\dot{\phi}^y\dot{\phi}^z + \tfrac{3}{2}e^{2U}V_{\rm bh}(\phi, B) = 0\,, \tag{2.53}$$

[12]The question remains whether the $H_I$ must be harmonic for all extremal black holes in all models. In section 3.2 we give a proof for static, spherically symmetric black holes of the $STU$ model.

where we introduced the *fake black hole potential*

$$V_{\rm bh}(\phi,B)\equiv -a^{IJ}\,B_IB_J=-\mathcal{Z}_{\rm e}^2(\phi,B)-3\,\partial_x\mathcal{Z}_{\rm e}(\phi,B)\partial^x\mathcal{Z}_{\rm e}(\phi,B)\,. \tag{2.54}$$

This means that the equations of motion will be satisfied for all these configurations if and only if the fake black-hole potential is equal to the true one:

$$V_{\rm bh}(\phi,B)\;=\;V_{\rm bh}(\phi,q)\,. \tag{2.55}$$

Observe that this equation is, up to an overall factor, nothing but the Hamiltonian constraint (2.29). Consequently, in the extremal case, if one finds values of $B_I$ (we could call them attractor values, by transfering the notion from $\phi_{\rm h}(B)$) that extremize the (genuine) black hole potential and satisfy the Hamiltonian constraint, then one has a solution of all the equations of motion.

The derivation of flow equations for extremal black holes presented here differs from their previous presentation [29, 30, 31] in two aspects: Firstly, no specific form of the relation between the fake and actual charges is assumed here, instead the derivation is based on the assumption of harmonicity of the variables $H_I$.[13] Secondly, expressing the black hole potential directly by harmonic functions makes it possible to determine the fake charges by extremization.

### 2.4 Non-extremal black holes

In the simple model studied in ref. [2] it was found that, as in the 4-dimensional case considered in ref. [12], the non-extremal black-hole solutions of that model as functions of the variables $H_I(\rho)$ are identical to the extremal ones. The difference is that, now, the functions $H_I(\rho)$ are no longer harmonic (*i.e.* linear in $\rho$) but have the general form

$$H_I(\rho)=A_I\cosh(\mathcal{B}\rho)+\frac{B_I}{\mathcal{B}}\sinh(\mathcal{B}\rho)\,, \tag{2.56}$$

for some integration constants $A_I$ and $B_I$ that a priori could be different from those in the extremal ansatz (2.27) and have to be determined by solving the equations of motion and by imposing the normalization of the physical fields at spatial infinity.

It is interesting to see whether the non-extremal black-hole solutions of more general models also share this generic form. We start by imposing the asymptotic boundary conditions on the fields, using the generic definition for the physical scalars eq. (2.31). It is easy to see that, again, asymptotic flatness implies eq. (2.30) and that the integration constants $A_I$ are given by eq. (2.35), so they are actually the same as in the extremal ansatz. Therefore we only have to determine the $B_I$ plus the non-extremality parameter $\mathcal{B}$ by imposing the equations of motion.

We will also need the definition of the mass, which is given again by eqs. (2.38) and (2.39), and the expressions for the horizon hyperarea and for the values of the scalars on the outer and inner horizon horizon, which are now

$$\varphi_I^{\pm}=\frac{B_I^{\pm}}{B_0^{\pm}}\,, \tag{2.57}$$

$$\frac{\mathcal{A}_{\pm}}{2\pi^2}=\mathsf{W}(B^{\pm})/2\,. \tag{2.58}$$

where we have defined the shifted coefficients (“dressed charges” [13])

$$B_I^{\pm}\equiv B_I\pm\mathcal{B}A_I\,. \tag{2.59}$$

[13] In four dimensions a generalization of first-order equations that makes neither assumption was made in [32].

Now, first of all, observe that the functions of eq. (2.56) satisfy

$$\ddot{H}_I = \mathcal{B}^2 H_I\,, \tag{2.60}$$

and then, substituting in eq. (2.26) and using the homogeneity properties of W, we find that it is identically satisfied. Substituting into the Hamiltonian constraint and the equations of motion and using the same properties we can rewrite them in the form

$$\partial^I \partial^J \log \mathsf{W}(H)\left(B_I^- B_J^+ - q_I q_J\right) = 0\,, \tag{2.61}$$

$$\partial^K \partial^I \partial^J \log \mathsf{W}(H)\left(B_I^- B_J^+ - q_I q_J\right) = 0\,. \tag{2.62}$$

In the near-horizon limits, these equations, upon use of the formula (2.58) for the area of the inner and outer horizons, lead to the following relations

$$\frac{\mathcal{A}_\pm}{2\pi^2} = \left[-\left(1 \mp \tfrac{4}{3}\mathcal{B} A_I \partial^I \log \mathsf{W}\right)^{-1} V_{\rm bh}\right]^{3/4}(B^\pm)\,, \tag{2.63}$$

$$\partial^K V_{\rm bh}(B^\pm) = \pm 6\left(\frac{\mathcal{A}_\pm}{2\pi^2}\right)^{4/3} \mathcal{B} A_I \left[\partial^I \partial^K \log \mathsf{W} + \frac{2}{3}\partial^I \log \mathsf{W}\, \partial^K \log \mathsf{W}\right](B^\pm)\,, \tag{2.64}$$

which generalize eqs. (2.43) and (2.44) to the non-extremal case. Since the right-hand side of eq. (2.64) does not vanish in general for non-extremal black holes, we find that, in general, the values of the scalars on the horizon do not extremize the black-hole potential. In section 3.1 we are going to study a class of non-extremal black holes for which the right-hand side of eq. (2.64) does vanish, though. We will be able to give the general form of this class of solutions for any model of $N = 2, d = 5$ supergravity.

For models with diagonal $\partial^I \partial^J \log \mathsf{W}$ (which, given that in $N = 2$ supergravity the polynomial $\mathcal{V}$ must be homogeneous of degree 3, comprise only two models, (apart from minimal supergravity): $STU$, discussed in the next section, and $ST^2$) the equations of motion (2.62) can be solved by [13]

$$B_I = \pm\sqrt{q_I^2 + \mathcal{B}^2 A_I^2} \quad \text{(no summation)}, \tag{2.65}$$

which completely determines the dressed charges, and thus the values of scalars on both horizons, in terms of physical charges, the asymptotic values of the scalars, eq. (2.33) and the non-extremality parameter:

$$B_I^- = \pm\sqrt{q_I^2 + (\mathcal{B} A_0 \varphi_{I\infty})^2} - \mathcal{B} A_0 \varphi_{I\infty}\,, \qquad B_I^+ = \pm\sqrt{q_I^2 + (\mathcal{B} A_0 \varphi_{I\infty})^2} + \mathcal{B} A_0 \varphi_{I\infty}\,, \tag{2.66}$$

These expressions reduce to ($\pm$ absolute values of) the actual charges when $\mathcal{B} \to 0$. As expected, due to the dependence on $\varphi_{I\infty}$ there is no attractor mechanism in the proper sense, but from $B_I^- B_I^+ = q_I^2$ we can make an new observation that the extremal attractor value of a scalar is the geometric mean of the non-extremal horizon values:

$$(\varphi^x_{\rm h})^2 = \varphi^x_- \varphi^x_+ \quad \text{(no summation)}. \tag{2.67}$$

## 2.5 First-order flow equations for non-extremal black holes

The derivation leading to eq. (2.49) can be followed straightforwardly with a small variation: instead of the coordinate $\rho$, we need a new coordinate $\hat\rho$ which is defined by[14]

$$\hat\rho \equiv \frac{\sinh(\mathcal{B}\rho)}{\mathcal{B}\cosh(\mathcal{B}\rho)}\,, \qquad \text{so that} \qquad \cosh(\mathcal{B}\rho) = \frac{1}{\sqrt{1 - \mathcal{B}\hat\rho^2}} \equiv f(\hat\rho)\,, \tag{2.68}$$

[14] This choice, like the parametrization of $H_I$, is not unique. One could, for instance, take $\hat\rho = e^{-2\mathcal{B}\rho}$. An advantage of the $\tanh$ parametrization is that the asymptotic values of scalars are still governed only by $A_I$, as in the extremal case.

which means that the ansatz for $H_I$ in eq. (2.56) now becomes the “almost extremal form”

$$H_I = f(\hat{\rho})\,(A_I + B_I\hat{\rho}) = f(\hat{\rho})\,\hat{H}_I\,. \tag{2.69}$$

Taking into account the above expression in the derivation leading to eq. (2.49) we find that

$$\frac{\partial e^{-\hat{U}}}{\partial\hat{\rho}} = \mathcal{Z}_{\rm e}(\phi, B)\,, \tag{2.70}$$

where we introduced $\hat{U} = U + \log(f)$. The hatted variables still satisfy $e^{\hat{U}}\hat{H}_I = e^U H_I = h_I$.

The analog of eq. (2.51) can be seen to be

$$\frac{\partial\phi^x}{\partial\hat{\rho}} = -3\,e^{\hat{U}}\partial^x\mathcal{Z}_{\rm e}(\phi, B)\,. \tag{2.71}$$

Since eqs. (2.70) and (2.68) have the same functional form as eqs. (2.49) and (2.51), we immediately see that they lead to the FGK equations of motion, albeit with respect to the new coordinate $\hat{\rho}$ and the new function $\hat{U}$,[15] *i.e.*

$$\partial^2_{\hat{\rho}}\hat{U} = -e^{2\hat{U}}V_{\rm bh}(\phi, B)\,, \tag{2.72}$$

$$\partial^2_{\hat{\rho}}\phi^x + \Gamma_{yz}{}^x\partial_{\hat{\rho}}\phi^y\partial_{\hat{\rho}}\phi^z = -\tfrac{3}{2}e^{2\hat{U}}\partial^x V_{\rm bh}(\phi, B)\,. \tag{2.73}$$

One can then ask oneself what the equivalent of eq. (2.55) is. To this end we shall rewrite the FGK-equation for $U$ in the $\hat{\rho}$-coordinate and use eq. (2.70) to get rid of a term linear in $\partial_{\hat{\rho}}U$:

$$-\mathcal{B}^2 + 2\mathcal{B}^2\hat{\rho}\,e^{\hat{U}}\mathcal{Z}_{\rm e}(\phi, B) = e^{2\hat{U}}\left(f^{-2}\,V_{\rm bh}(\phi, B) - V_{\rm bh}(\phi, q)\right). \tag{2.74}$$

This equation is the Hamiltonian constraint written in the new coordinates $\hat{\rho}$. To see this we need

$$e^{-U}\mathcal{Z}_{\rm e}(\phi, B) = \tfrac{2}{3}\,B_I\partial^I\log(\mathsf{W})\,. \tag{2.75}$$

Performing the same operation on the FGK-equation for the scalar fields, we find that after using eq. (2.71)

$$4\mathcal{B}^2\,\hat{\rho}\,e^{\hat{U}}\,\partial^x\mathcal{Z}_{\rm e}(\phi, B) = e^{2\hat{U}}\left(f^{-2}\,\partial^x V_{\rm bh}(\phi, B) - \partial^x V_{\rm bh}(\phi, q)\right). \tag{2.76}$$

The extra factor of 2 on the left-hand side of the above equation compared to eq. (2.74) is surprising, but correct; indeed, differentiating eq. (2.74) with respect to $\hat{\rho}$ and using the flow equations (2.70) and (2.71) we find that

$$0 = \partial_{\hat{\rho}}\phi^y\,g_{yx}\left[4\mathcal{B}^2\hat{\rho}\,e^{\hat{U}}\partial^x\mathcal{Z}_{\rm e} - e^{2\hat{U}}\left(f^{-2}\,\partial^x V_{\rm bh}(B) - \partial^x V_{\rm bh}(q)\right)\right]. \tag{2.77}$$

This implies that eq. (2.74) is a constant if eq. (2.76) is satisfied, whence we can evaluate it at spatial infinity, *i.e.* $\rho = 0$ and also $\hat{\rho} = 0$. This gives

$$V_{\rm bh}(\phi_\infty, q) - V_{\rm bh}(\phi_\infty, B) = \mathcal{B}^2 e^{-2U_\infty} = \mathcal{B}^2 \tag{2.78}$$

by asymptotic flatness.

The flow (2.71) terminates at the horizon ($\hat{\rho} \to 1/\mathcal{B}$). In the extremal case ($\hat{\rho} \to \infty$) or in the non-extremal case with constant scalars, since $\partial_{\hat{\rho}}\phi^x\big|_{\rm h} = 0$, the horizon value of the scalars will be

[15] Note that from $U = U(H)$ and the scaling properties of $\mathsf{W}$, we can see that $\hat{U} = U(\hat{H})$.

determined by the location of the fixed point (attractor) $\partial_x \mathcal{Z}_{\rm e}(B) = 0$. For a generic non-extremal solution the horizon value will be attained in a finite $\hat{\rho}$, before a fixed point is reached.[16] We can still evaluate the relevant equations at the horizon to find

$$-\mathcal{B}^2 + 2\mathcal{B}\, e^{\hat{U}_{\rm h}} \mathcal{Z}_{\rm e}(\phi_{\rm h}, B) = -e^{2\hat{U}_{\rm h}} V_{\rm bh}(\phi_{\rm h}, q)\,, \tag{2.79}$$

$$4\mathcal{B}\, e^{-\hat{U}_{\rm h}} \partial_x \mathcal{Z}_{\rm e}(\phi_{\rm h}, B) = -\partial_x V_{\rm bh}(\phi_{\rm h}, q)\,. \tag{2.80}$$

# 3 Example black-hole solutions

In this section we illustrate the use of the H-FGK formalism with some examples, two of which have supersymmetric and non-supersymmetric attractors and flat directions. The third one is a generic class of solutions whose main characteristic is that the physical scalars are constants, and as such are a generalization of the doubly extremal black holes.

## 3.1 Non-extremal black holes with constant scalars

When all the scalar fields of an extremal black-hole solution are constant, it is known as a doubly extremal black hole. It is natural to consider its non-extremal generalizations, *i.e.* non-extremal black holes with constant scalars. For the general ansatz (2.56) and the generic parametrization (2.31) of the physical scalars, this condition requires that

$$\varphi_I = \frac{A_I}{A_0} = \frac{B_I}{B_0} = \frac{B_I^{\pm}}{B_0^{\pm}}\,, \tag{3.1}$$

and we can write

$$H_I = A_I H\,, \qquad H \equiv \cosh(\mathcal{B}\rho) + \frac{B_0}{A_0 \mathcal{B}} \sinh(\mathcal{B}\rho)\,, \tag{3.2}$$

so we have

$$\mathsf{W}(H) = 2H^{3/2}\,. \tag{3.3}$$

The metric is, as expected, that of the 5-dimensional Reissner-Nordström black hole in all cases.

The only integration constants that need to be found are $B_0$ and $\mathcal{B}$. It is convenient to introduce in the problem the mass parameter, given by eq. (2.38). In this case, it is just

$$M = \frac{B_0}{A_0}\,. \tag{3.4}$$

Then the Hamiltonian constraint and the equations of motion take the form

$$\partial^I \partial^J \log \mathsf{W}(A) \left[(M^2 - \mathcal{B}^2) A_I A_J - q_I q_J\right] = 0\,, \tag{3.5}$$

$$\partial^K \partial^I \partial^J \log \mathsf{W}(A) \left[(M^2 - \mathcal{B}^2) A_I A_J - q_I q_J\right] = 0\,, \tag{3.6}$$

and are solved if

$$\mathcal{B}^2 - M^2 - V_{\rm bh}(A, q) = 0\,, \tag{3.7}$$

$$\partial^K V_{\rm bh}(A, q) = 0\,. \tag{3.8}$$

[16] This in effect is the argument given in ref. [33] as to why the attractor mechanism cannot work for non-extremal black holes; other arguments are given in ref. [34]. If we were to extend $\hat{\rho}$ beyond $1/\mathcal{B}$ to infinity, the values of scalars would be again determined by the ratios of $B_I$ (this occurs between the horizons [13]), the values of $B_I$, however, now depend on the asymptotic boundary conditions.

The first equation is just the general Bogomol'nyi bound for constant scalars (vanishing scalar charges) and the second, owing to the scale invariance of $V_{\rm bh}(H)$ tells us that the scalars are not affected by the non-extremality parameter and everywhere take the values that extremize the black-hole potential, which are the same as in the extremal case with the same electric charges. The value of the black-hole potential (in particular, on the horizon) is also the same as in the extremal case with the same electric charges. Notice that this implies that $V_{\rm bh}(A,q)$ is a function of the charges $q$ only.

Using this information in eq. (2.63) and invoking the properties of W, we also obtain an expression that relates the entropy to the entropy of the extremal black hole with the same electric charges:

$$\frac{\mathcal{A}_\pm}{2\pi^2} = \left(-\frac{M\pm\mathcal{B}}{M\mp\mathcal{B}}\, V_{\rm bh}(B^\pm)\right)^{3/4}, \tag{3.9}$$

Since $V_{\rm bh}(B^\pm) = V_{\rm bh}(A)$, combining this expression with the Bogomol'nyi bound we get the well-known formula

$$\frac{\mathcal{A}_\pm}{2\pi^2} = (M\pm\mathcal{B})^{3/2}\ , \tag{3.10}$$

which also admits the suggestive expression

$$\frac{\mathcal{A}_\pm}{2\pi^2} = \mathcal{Z}_{\rm e}(\varphi_\infty, B^\pm)^{3/2}\,, \tag{3.11}$$

and leads to the suggestive relation

$$\frac{\mathcal{A}_+}{2\pi^2}\frac{\mathcal{A}_-}{2\pi^2} = (M^2-\mathcal{B}^2)^{3/2} = [-V_{\rm bh}(A,q)]^{3/2} = \left(\frac{\mathcal{A}_{\rm ext}}{2\pi^2}\right)^2\,, \tag{3.12}$$

where, as we stressed above, $V_{\rm bh}(A,q)$ is moduli-independent and $\mathcal{A}_{\rm ext}$ is hyperarea of the extremal black hole with the same charges. We will refer to this property in what follows as the *geometrical mean property*[17].

Summarizing, the solutions of this class, for any model, are obtained by finding first the values (determined up to a common factor) of the $B_I^\pm$ that extremize the potential $\partial^K V_{\rm bh}(B^\pm) = 0$. The scalars are then given by $\varphi_I = B_I^\pm/B_0^\pm$, which, through eq. (2.36), dictates the constants $A_I$ for these values of the scalars. The non-extremality parameter is established by eq. (3.7), the metric function is $e^{-U/2} = H$ with $H$ as in eq. (3.2), and the mass of this black hole is found from eq. (3.4).

### 3.1.1 Constant-scalar black holes from the flow equations

As one can see from eq. (2.71), constant scalars around a black hole satisfy

$$\partial_x \mathcal{Z}_{\rm e}(B) = 0 \qquad \Rightarrow \qquad \partial_x V_{\rm bh}(\phi, B) = 0\,. \tag{3.13}$$

We can then use eq. (2.70) to obtain

$$e^{-\hat{U}} = 1 + \mathcal{Z}_{\rm e}(B)\hat{\rho}\,, \tag{3.14}$$

where we already imposed asymptotic Minkowskianity. It is also evident that the mass of the solution is given by $M = \mathcal{Z}_{\rm e}(B)$. Plugging the conditions in eq. (3.13) into eq. (2.76) we see that

$$\partial_x V_{\rm bh}(\phi, q) = 0\,, \tag{3.15}$$

[17] A proof for the charged, rotating, asymptotically-flat or anti-de-Sitter black-hole solutions of a wide class of theories (which does not include those we are considering here) has been given in [35]. Earlier, less general results, were found in [36, 37, 38, 39, 40]. A related result valid for horizons of arbitrary topology has been recently found in [41].

which is the analog of eq. (3.8). Eq. (3.7) can be derived immediately from the Hamiltonian constraint (2.78).

Eq. (3.13) says that, in terms of the fake charges, the constant scalars of a non-supersymmetric solution have the same form as the scalars of the supersymmetric extremal solution in terms of the real charges, whereas eq. (3.15) fixes the scalars directly in terms of $q$'s.

## 3.2 The $STU$ model revisited

The $STU$ model in five dimensions is defined by

$$\mathcal{V}(h^{\cdot}) = h^0 h^1 h^2 = 1\,. \tag{3.16}$$

The corresponding unconstrained function in the H-FGK formalism is

$$\mathsf{V}(\tilde{H}) = \tilde{H}^0 \tilde{H}^1 \tilde{H}^2\,. \tag{3.17}$$

The tilded variables are given in terms of the untilded ones by

$$\tilde{H}^0 = \sqrt{\frac{3H_1H_2}{H_0}}\,, \qquad \tilde{H}^{1,2} = \frac{3H_{2,1}}{\tilde{H}^0}\,, \tag{3.18}$$

so

$$\mathsf{W}(H) = 2\,\mathsf{V}(H) = 2\sqrt{3^3 H_0 H_1 H_2}\,. \tag{3.19}$$

This potential contains all the information that we need to find and construct all the black-hole solutions of the model.

Due to the special form of $\mathsf{W}$, the equations of motion (2.24) are completely separated and read

$$H_I \ddot{H}_I - \dot{H}_I^2 + q_I^2 = 0 \quad \text{(no summation)}\,. \tag{3.20}$$

These equations can be integrated explicitly, with the general solution being of the form

$$H_I = a_I \cosh(\varepsilon_I \rho) + b_I \sinh(\varepsilon_I \rho)\,. \tag{3.21}$$

The Hamiltonian constraint, eq. (2.25), then imposes the condition $\sum_I \varepsilon_I^2 = 3\mathcal{B}^2$. A further constraint arises due to the fact that we are interested in building black holes as expressed by eq. (2.7), which implies that $\sum_I \varepsilon_I = 3\mathcal{B}$. In addition, we should also have scalar fields that are regular on the horizon. We can do this by imposing the condition that the $\varphi_I$ be regular there, which also ensures the regularity of the physical scalars $\phi^x$ on the horizon. Clearly this means that $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \mathcal{B}$, reducing the general solution to the ansatz (2.56). In other words, in the $STU$ model, all black-hole solutions of the type considered here must be described by this ansatz.

### 3.2.1 Extremal solutions

In the limit $\mathcal{B} \to 0$ the ansatz (2.56) reduces to eq. (2.27) and it follows from the argument above that all extremal solutions to the $STU$ model (at least of the kind we are analyzing) will be described by harmonic functions.

To determine their coefficients, let us first analyze the critical points of the black-hole potential $V_{\text{bh}}(H,q)$, which for the $STU$ model takes the form

$$V_{\text{bh}}(H,q) = \tfrac{2}{3}\,(\mathsf{W}/2)^{4/3}\,\partial^I\partial^J \log \mathsf{W}\, q_I q_J = -3(H_0H_1H_2)^{2/3} \sum_I \left(\frac{q_I}{H_I}\right)^2\,. \tag{3.22}$$

The equations $\partial^K V_{\rm bh}(H,q)\big|_{\rm h} = 0$ are solved by

$$(B_I)^2 = \alpha^2 (q_I)^2\,, \tag{3.23}$$

where $\alpha$ is an arbitrary constant, resulting from the scale invariance of the black-hole potential as a function of the $H$'s. This constant does not affect the attractor points of the physical scalars, which are given by quotients of $H$'s.

The above solutions correspond to

$$B_I = s_I q_I\,, \tag{3.24}$$

where the signs $s_I = \pm 1$ can, in principle, be chosen at will and are independent of the electric charges. Each choice of signs corresponds to a different kind of solution that may or may not (we will carefully look into this point) describe several signs of the charges. The reality and regularity of the metric and scalar fields will impose certain restrictions on the possible signs, though. First of all, the reality of $\mathsf{W}(B)$ requires that

$$\beta\,\text{sgn}(q_0)\,\text{sgn}(q_1)\,\text{sgn}(q_2) = +1\,, \qquad \beta \equiv s_0 s_1 s_3\,. \tag{3.25}$$

The attractor values for the physical scalars, chosen as in eq. (2.31) for $x = 1, 2$, on the horizon are

$$\varphi_{x\,\rm h} = s_0 s_x q_x / q_0\,. \tag{3.26}$$

In terms of these scalars the sections are given by

$$h_0 = \frac{1}{3(\varphi_1\varphi_2)^{1/3}}\,, \qquad h_1 = \frac{\varphi_1}{3(\varphi_1\varphi_2)^{1/3}}\,, \qquad h_2 = \frac{\varphi_2}{3(\varphi_1\varphi_2)^{1/3}}\,, \tag{3.27}$$

$$h^0 = (\varphi_1\varphi_2)^{1/3}\,, \qquad h^1 = \frac{(\varphi_1\varphi_2)^{1/3}}{\varphi_1}\,, \qquad h^2 = \frac{(\varphi_1\varphi_2)^{1/3}}{\varphi_2}\,. \tag{3.28}$$

Observe that the scalars can be positive or negative but not zero. This means that the theory has four branches[18] that can be labeled by the four possible combinations of the two signs of the scalars: $\sigma_x \equiv \text{sgn}(\varphi_x)$. In terms of unconstrained scalars $\phi_x$ (customarily called for this model $S$ and $T$) we would have $\varphi_x \sim \sigma_x e^{\phi_x}$. Since, in a regular solution for a given branch $\sigma_1, \sigma_2$, the above scalars will have the same sign everywhere and in particular on the horizon, from eq. (3.26) we find that admissible regular extremal solutions in the branch $\sigma_x$ satisfy besides eq. (3.25) also

$$s_0\, s_x = \sigma_x\,\text{sgn}(q_0)\,\text{sgn}(q_x)\,. \tag{3.29}$$

This condition ensures that $\text{sgn}(B_I) = \text{sgn}(A_I)$ for all $I$, so the functions $H_I$ do not vanish for any positive value of the radial coordinate $\rho$, which is another condition for regularity of the solution. In any given branch $\sigma_x$, for any of the 8 possible choices of signs of the charges, there is always a choice of $s_I$ that allows us to have a regular extremal solution.

To find out which of these solutions are supersymmetric in each branch, we need to extremize the central charge, which is given by

$$\mathcal{Z}_{\rm e}(\varphi, q) = (\varphi_1\varphi_2)^{-2/3}\left(q_0\varphi_1\varphi_2 + q_1\varphi_2 + q_2\varphi_1\right)\,. \tag{3.30}$$

The extrema correspond to the horizon values

$$\varphi_{x\,\rm h}^{\rm SUSY} = q_x/q_0\,. \tag{3.31}$$

[18] The scalar manifold has to be covered by four coordinate patches.

| $\mathrm{sgn}(q_0)$ | $\mathrm{sgn}(q_1)$ | $\mathrm{sgn}(q_3)$ | $s_0$ | $s_1$ | $s_2$ | SUSY |
|---|---|---|---|---|---|---|
| $+$ | $+$ | $+$ | $-$ | $-$ | $+$ | no |
| $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | yes |
| $+$ | $-$ | $+$ | $-$ | $+$ | $+$ | no |
| $+$ | $-$ | $-$ | $-$ | $+$ | $-$ | no |
| $-$ | $+$ | $+$ | $+$ | $-$ | $+$ | no |
| $-$ | $+$ | $-$ | $+$ | $-$ | $-$ | no |
| $-$ | $-$ | $+$ | $+$ | $+$ | $+$ | yes |
| $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | no |

Table 1: In the last three columns of this table we give the choices of signs $s_I$ necessary to have regular solutions for each of the combinations of signs of the electric charges given in the first three columns in the branch $\sigma_1 = 1,\, \sigma_2 = -1$. The supersymmetric configurations of this branch are those in the second and seventh rows.

Comparing with eq. (3.26) we find that only the cases

$$\mathrm{sgn}(q_0)\ \mathrm{sgn}(q_x) = \sigma_x\,, \tag{3.32}$$

are supersymmetric in the branch $\sigma_x$. This corresponds to two possible sign configurations for each of the four branches so all charge configurations are supersymmetric in some branch, as in the case considered in ref. [2].

Combining the choices of $s_I$ with the signs of the charges, all the solutions can be written in a unified way in terms of the harmonic functions

$$H_I = \frac{\mathrm{sgn}(q_0)\ \mathrm{sgn}(q_1)\ \mathrm{sgn}(q_2)}{\mathrm{sgn}(q_I)} \left( \frac{4^{1/3}}{3} \left| \frac{\varphi_{I\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}} \right| + |q_I|\rho \right)\,, \tag{3.33}$$

which are manifestly non-vanishing for positive values of $\rho$ and are valid for all four branches and all eight combinations of the signs of the charges. Furthermore, in all cases and for all choices of the signs of the charges (contrary to what is stated in ref. [16]), the metric function $e^{-\frac{3}{2}U} = \mathsf{W}(H)/2$ is real and regular. The entropy, given by eq. (2.41), takes the explicit form

$$\frac{\mathcal{A}}{2\pi^2} = \tfrac{1}{2}\sqrt{3^3|q_0 q_1 q_2|}\,, \tag{3.34}$$

and the mass, given by eq. (2.39) becomes

$$M = (\varphi_{1\,\infty}\varphi_{2\,\infty})^{-2/3}\Big(|\varphi_{1\,\infty}\varphi_{2\,\infty}\, q_0| + |\varphi_{2\,\infty}\, q_1| + |\varphi_{1\,\infty}\, q_2|\Big)\,, \tag{3.35}$$

which is always positive. It coincides with the supergravity central charge at infinity only for certain signs of the charges that depend on the branch considered, as we have explained before.

### 3.2.2 Non-extremal solutions

For the present model the ansatz (2.56) is the general solution to the equations of motion (2.62) or (3.20) and they reduce to the following relation between the parameters:

$$B_I^2 = \mathcal{B}^2 A_I^2 + q_I^2\,. \tag{3.36}$$

Since the integration constants $A$ are, according to the general arguments, given by

$$A_I = \frac{4^{1/3}}{3} \frac{\varphi_{I\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}}\,, \tag{3.37}$$

the above equations immediately give the complete solution

$$B_I = s_I \sqrt{q_I^2 + \mathcal{B}^2 A_I^2}\,, \tag{3.38}$$

where we have to choose the signs $s_I$ so that the functions $H_I$ do not vanish for any value of $\rho > 0$, *i.e.* so that $\mathrm{sgn}(B_I) = \mathrm{sgn}(A_I)$ (assuming $\mathcal{B} > 0$):

$$s_I = \frac{\mathrm{sgn}(\varphi_{0\,\infty})\ \mathrm{sgn}(\varphi_{1\,\infty})\ \mathrm{sgn}(\varphi_{2\,\infty})}{\mathrm{sgn}(\varphi_{I\,\infty})}\,. \tag{3.39}$$

The regularity of the metric translates into manifest positivity of the mass, given by eq. (2.39) and, explicitly, by

$$\begin{aligned} M = {} & \sqrt{\left[(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3} q_0\right]^2 + \mathcal{B}^2 \left[(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3} A_0\right]^2} \\ & + \sqrt{\left[\frac{\varphi_{2\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}} q_1\right]^2 + \mathcal{B}^2 \left[\frac{\varphi_{2\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}} A_1\right]^2} \\ & + \sqrt{\left[\frac{\varphi_{1\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}} q_2\right]^2 + \mathcal{B}^2 \left[\frac{\varphi_{1\,\infty}}{(\varphi_{1\,\infty}\varphi_{2\,\infty})^{1/3}} A_2\right]^2}\,. \end{aligned} \tag{3.40}$$

The non-extremality parameter can be solved in terms of the mass, asymptotic values of the scalars and charges by solving a quartic algebraic equation in $\mathcal{B}^2$, but the expression is too complicated to be useful.

The hyperareas of the horizons, given by the general formula (2.58), take the form

$$\frac{\mathcal{A}_\pm}{2\pi^2} = \frac{1}{2}\left[3^3 \prod_I \left(\sqrt{q_I^2 + \mathcal{B}^2 A_I^2} \pm \mathcal{B}|A_I|\right)\right]^{1/2}\,. \tag{3.41}$$

We can see explicitly that not only the values of scalars, as mentioned earlier, but also the entropies $S_\pm = \mathcal{A}_\pm/4$ satisfy the geometric mean property:

$$S_- S_+ = S^2\,, \tag{3.42}$$

where the mean value is that of the extremal black hole.

### 3.3 Models of the generic Jordan family

The models of the reducible Jordan sequence are defined by

$$\mathcal{V}(h^\cdot) = h^0 \eta_{ij} h^i h^j = 1\,, \qquad i = 1, \ldots, n\,, \tag{3.43}$$

where $(\eta_{ij}) = \mathrm{diag}(- + \cdots +)$ and the associated potential in the H-FGK formalism takes the form

$$\mathsf{V}(\tilde{H}) = \tilde{H}^0 \tilde{H}^2\,, \tag{3.44}$$

where we have defined

$$\tilde{H}^2 \equiv \tilde{H}^i \tilde{H}_i \equiv \eta_{ij} \tilde{H}^i \tilde{H}^j \equiv \tilde{H} \cdot \tilde{H} \,. \tag{3.45}$$

The relation between tilded and untilded variables is

$$\tilde{H}^0 = \frac{1}{2} \sqrt{\frac{3H^2}{H_0}} \,, \qquad \tilde{H}^i = H^i \sqrt{\frac{3H^0}{H^2}} \,, \tag{3.46}$$

so

$$\mathsf{W}(H) = 2\mathsf{V}(H) = \sqrt{3^3 H_0 H^2} \,, \tag{3.47}$$

The non-vanishing components of the Hessian of $\log \mathsf{W}$ are

$$\partial^0 \partial^0 \log \mathsf{W} = -\frac{1}{2H_0^2} \,, \qquad \partial^i \partial^j \log \mathsf{W} = \frac{\eta^{ij} H^2 - 2H^i H^j}{(H^2)^2} \,. \tag{3.48}$$

### 3.3.1 Extremal solutions

There are two kinds of critical loci of the black-hole potential: the discrete points

$$B_i = s q_i \,, \quad s^2 = +1 \,, \tag{3.49}$$

$$\frac{B^2}{B_0^2} - \frac{q^2}{q_0^2} = 0 \,, \tag{3.50}$$

including those that correspond to supersymmetric black holes, and the $(n-1)$-dimensional space described by the constraint

$$B \cdot q = 0 \,, \tag{3.51}$$

$$\frac{B^2}{B_0^2} + \frac{q^2}{q_0^2} = 0 \,, \tag{3.52}$$

which gives rise to non-supersymmetric solutions. We focus on the latter since we expect the scalars on the horizon to depend on the values of the scalars at infinity.

The constraint eq. (3.51) is solved in a general way by

$$B_i = \alpha \left[ (C \cdot q) q_i - q^2 C_i \right] \,, \tag{3.53}$$

for some constants $C_i$ that are defined only up to shifts proportional to the charges $q_i$ and up to a normalization constant $\alpha$. The limit of the Hamiltonian constraint (2.29) on the horizon is solved by

$$B^2 = -q^2 \,, \tag{3.54}$$

and then the second condition eq. (3.52) determines the integration constants $B_0$

$$B_0^2 = q_0^2 \qquad \Rightarrow \qquad B_0 = s_0 q_0 \,, \quad s_0^2 = +1 \,. \tag{3.55}$$

Plugging the general solution for $B_i$ into $B^2 = -q^2$ we find that

$$\alpha^2 = \left[ (C \cdot q)^2 - C^2 q^2 \right]^{-1} \qquad \Rightarrow \qquad B_i = s \frac{(C \cdot q) q_i - q^2 C_i}{\sqrt{(C \cdot q)^2 - C^2 q^2}} \,, \quad s^2 = +1 \,, \tag{3.56}$$

so the coefficients $B_i$ have a highly non-linear dependence on the charges, something that could make us think that the $H_i$ might be highly non-linear functions of harmonic functions. Evidently, we have assumed from the onset the harmonicity of these variables and our challenge is to prove that the ansatz solves all the equations of motion for this moduli space of non-supersymmetric attractors.

The asymptotic limit of the Hamiltonian constraint (2.29) is solved if the constants $C_i$ are proportional to the $A_i$, whose value is known. We take the proportionality constant to be 1 and then it becomes just a matter of calculation to see that eq. (2.29) is satisfied everywhere. The $K = 0$ component of the equations of motion (2.28) is trivially satisfied and, again, it is a matter of calculation to check that the $K = k$ components, which have the form

$$\left(3\eta^{(ij}H^{k)}H^2 - 4H^iH^jH^k\right)(B_iB_j - q_iq_j) = 0\,, \tag{3.57}$$

are also identically satisfied for

$$B_i = s\frac{(A\cdot q)q_i - q^2A_i}{\sqrt{(A\cdot q)^2 - A^2q^2}}\,, \quad s^2 = +1\,. \tag{3.58}$$

### 3.3.2 Non-extremal solutions

It is not too difficult to extend the extremal solutions to the non-extremal regime using the ansatz (2.56). For that purpose it is convenient to introduce the mass parameter. According to the general expression (2.39), it is given by

$$M = \frac{2}{3}\left(\frac{B_0}{2A_0} + \frac{A\cdot B}{A^2}\right)\,, \tag{3.59}$$

and we can use this formula to express the combination $A\cdot B$ as

$$A\cdot B = \frac{A^2}{2}\left(3M - \frac{B_0}{A_0}\right)\,. \tag{3.60}$$

All the terms in the right-hand side of this expression are known in terms of physical parameters except for $B_0$, but this constant can be found by solving the $K = 0$ equation of motion:

$$B_0 = s_0\sqrt{q_0^2 + \mathcal{B}^2A_0^2}\,, \tag{3.61}$$

so

$$A\cdot B = \frac{A^2}{2}\left(3M - \frac{s_0}{A_0}\sqrt{q_0^2 + \mathcal{B}^2A_0^2}\right)\,. \tag{3.62}$$

In order to keep the expressions as simple as possible, we will not replace $A\cdot B$ by its above value in what follows.

The $K = k$ equations of motion

$$\left(3\eta^{(ij}H^{k)}H^2 - 4H^iH^jH^k\right)\left(B_iB_j - \mathcal{B}^2A_iA_j - q_iq_j\right) = 0\,, \tag{3.63}$$

upon use of the Hamiltonian constraint

$$H^2(B^2 - \mathcal{B}^2A^2 - q^2) - 2\left[(H\cdot B)^2 - \mathcal{B}^2(H\cdot A)^2 - (H\cdot q)^2\right] = 0\,, \tag{3.64}$$

can be expanded in a finite number of powers of $\tanh \mathcal{B}\rho$ and, requiring that all the coefficients vanish, we get two equations:

$$A^k(B^2 - \mathcal{B}^2 A^2 - q^2) - 2\left[B^k(A\cdot B) - \mathcal{B}^2 A^k A^2 - q^k(A\cdot q)\right] = 0\,, \tag{3.65}$$

$$B^k(B^2 - \mathcal{B}^2 A^2 - q^2) - 2\left[B^k B^2 - \mathcal{B}^2 A^k (A\cdot B) - q^k(B\cdot q)\right] = 0\,. \tag{3.66}$$

It can be checked that these two equations imply the Hamiltonian constraint, therefore it is enough to solve only them. These equations contain two unknown combinations of integration constants: $B^2$ and $B\cdot q$, which can be found by multiplying the above equations by $A_k, B_k$ or $q_k$. We get

$$B^2 = \mathcal{B}^2 A^2 + q^2 + \frac{2}{A^2}\left[(A\cdot B)^2 - \mathcal{B}^2(A^2)^2 - (A\cdot q)^2\right], \tag{3.67}$$

$$B\cdot q = \frac{A\cdot B}{(A\cdot q)A^2}\left[q^2A^2 + (A\cdot B)^2 - \mathcal{B}^2(A^2)^2 - (A\cdot q)^2\right]. \tag{3.68}$$

Substituting these expressions in the two equations above we obtain two different equations for $B^k$ in terms of the known objects $(A\cdot B), \mathcal{B}, A^k, A^2, (A\cdot q)$:

$$B^k = \frac{[(A\cdot B)^2 - (A\cdot q)^2]A^k + A^2(A\cdot q)q^k}{A^2(A\cdot B)}\,, \tag{3.69}$$

$$B^k = \frac{(A\cdot B)\left\{A^2(A\cdot q)\mathcal{B}^2A^k + [q^2A^2 + (A\cdot B)^2 - \mathcal{B}^2(A^2)^2 - (A\cdot q)^2]q^k\right\}}{(A\cdot q)\left[q^2A^2 + (A\cdot B)^2 - (A\cdot q)^2\right]}\,. \tag{3.70}$$

These two solutions must be equal and one can see that this happens when the following condition is satisfied:

$$(A^2)^2(A\cdot B)^2\mathcal{B}^2 = [(A\cdot B)^2 - (A\cdot q)^2]^2 + A^2q^2[(A\cdot B)^2 - (A\cdot q)^2]\,. \tag{3.71}$$

This condition, on account of eq. (3.62), is an equation that involves $M, \mathcal{B}^2, A_I$ and $q_I$ and, in principle, may be used to express the non-extremality parameter as $\mathcal{B}^2(M, \varphi_{x\,\infty}, q_I)$.

# 4 H-FGK formalism for black-string solutions

In this section we will develop a formalism analogous to the one in section 2, but for obtaining string-like solutions; the derivation follows similar lines, the only difference being the identification of the new variables. Indeed, as one can see from refs. [26, 18, 24], the seed-functions for supersymmetric string-like solutions are not related to the $h_I$ as in the black hole, but rather to the $h^I$. As such, the formalism to be developed and illustrated in this section will be based on new variables $K^I$ and $\tilde{K}_I$, ($I = 0, \ldots, n$), which we define by

$$K^I \equiv e^{-U}h^I(\phi)\,. \tag{4.1}$$

By substituting this change of variables into the fundamental constraint of real special geometry and defining

$$\mathsf{V}(K) \equiv C_{IJK}K^IK^JK^K\,, \qquad \text{we find that} \qquad e^{-3U} = \mathsf{V}(K)\,. \tag{4.2}$$

We can then introduce the dual variables $\tilde{K}_I$ by

$$\tilde{K}_I = e^{-2U}h_I(\phi) \qquad \text{or equivalently} \qquad \tilde{K}_I = \tfrac{1}{3}\partial_I\mathsf{V}(K)\,, \tag{4.3}$$

but they will be used sparingly in this section.

The FGK formalism for black holes can be generalized to the case of branes [3]. The generic metric for 5-dimensional black-string solutions is

$$ds^2 = e^{U(\rho)-\mathcal{B}\rho}dt^2 - e^{U(\rho)+\mathcal{B}\rho}dy^2 - e^{-2U(\rho)}\left(\frac{\mathcal{B}^4}{\sinh^4(\mathcal{B}\rho)}d\rho^2 + \frac{\mathcal{B}^2}{\sinh^2(\mathcal{B}\rho)}d\Omega^2_{(2)}\right), \tag{4.4}$$

where $d\Omega^2_{(2)}$ denotes the round metric on the 2-sphere. Observe that the function $U$ in the above metric must satisfy $\lim_{\rho\to\infty}(U+\mathcal{B}\rho)=0$ in order for the metric to describe a black string with a horizon located at $\rho\to\infty$. If this condition is met, the near-horizon geometry for $\mathcal{B}\neq 0$ is a 2-dimensional Rindler space times $\mathbb{R}\times S^2$; in the extremal case, $\mathcal{B}=0$, the near-horizon geometry is $aDS_3\times S^2$ as usual.

We consider purely "magnetic" black string solutions, meaning that we take $F^I \sim \star_{(5)}(dv\wedge dt\wedge dx)$, the implication of which is that we can safely ignore the Chern-Simons term in the parent $d=5$ supergravity action and straightforwardly Hodge-dualize the $F^I$. The resulting action reads

$$\mathcal{I}_5 = \int_5 \sqrt{g}\left(R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2\cdot 3!}a^{IJ}G_{I\mu\nu\kappa}G_J^{\mu\nu\kappa}\right), \tag{4.5}$$

where $G_I = dB_I$. The resulting equations of motion for the above action are

$$R_{\mu\nu} = -\tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial_\nu\phi^y - \tfrac{1}{4}a^{IJ}\left(G_{I\mu\kappa\lambda}G_{J\nu}{}^{\kappa\lambda} - \tfrac{2}{9}\eta_{\mu\nu}G_I\cdot G_J\right). \tag{4.6}$$

Given the ansatz for the metric, the $B$ equation of motion is readily solved to give

$$G_I = \sqrt{3}\,e^{2U}a_{IJ}\,p^J dv\wedge dt\wedge dx\,, \tag{4.7}$$

where $p^I$ are the string charges and the $\sqrt{3}$ is inserted for convenience.

By simply substituting the ansatz into eq. (4.6) we find that

$$\ddot{U} = -e^{2U}a_{IJ}p^Ip^J\,, \tag{4.8}$$

$$\mathcal{B}^2 = \dot{U}^2 + \tfrac{1}{3}g_{xy}\dot{\phi}^x\dot{\phi}^y - e^{2U}a_{IJ}p^Ip^J\,. \tag{4.9}$$

We will not give the equations of motion for the scalars as the result should be obvious. The resulting FGK action can then be seen with the aid of eqs. (2.2) and (2.4) to be

$$\mathcal{I}[U,\phi^x] = \int d\rho\left(\dot{U}^2 + a_{IJ}\dot{h}^I\dot{h}^J - e^{2U}V_{\rm st} + \mathcal{B}^2\right), \tag{4.10}$$

where we have defined the *black string potential* as

$$V_{\rm st} \equiv -a_{IJ}\,p^Ip^J = -\mathcal{Z}_{\rm m} - 3\partial_x\mathcal{Z}_{\rm m}\partial^x\mathcal{Z}_{\rm m}\,, \tag{4.11}$$

where in the last step we introduced the *(magnetic) string central charge* $\mathcal{Z}_{\rm m} = h_Ip^I$; $\mathcal{B}$ is again a non-extremality parameter. In order to obtain the above action in the $K$-variables, we will need the straightforward identity

$$-\,3\,e^{2U}a_{IJ} = \partial_I\partial_J\log\mathsf{V} \equiv v_{IJ}\,, \tag{4.12}$$

which then enables us to write eq. (4.10) as

$$-\,3\,\mathcal{I}[K] = \int d\rho\left(v_{IJ}\big(\dot{K}^I\dot{K}^J + p^Ip^J\big) - 3\mathcal{B}^2\right). \tag{4.13}$$

The Hamiltonian constraint can be expressed as

$$\mathcal{H} \equiv v_{IJ}\big(\dot{K}^I\dot{K}^J - p^I p^J\big) + 3\mathcal{B}^2 = 0 \tag{4.14}$$

and the equations of motion derived from the effective action are

$$\partial_I v_{KL}\big(\dot{K}^K\dot{K}^L - K^K\ddot{K}^L - p^K p^L\big) = 0\,. \tag{4.15}$$

After contraction with $K^I$ and some minor manipulations they lead to

$$\ddot{K}^I\partial_I \log \mathsf{V} = 3\mathcal{B}^2\,, \tag{4.16}$$

which can be written as $\big(\ddot{K}_I - \mathcal{B}^2 K^I\big)\partial_I\mathsf{V} = 0$.

The resulting equations are very similar to the ones obtained in the black-hole case (*cf.* eqs. (2.25, 2.24, 2.26)), we thus make the ansatz of the same type as eq. (2.56),

$$K^I = A^I\cosh(\mathcal{B}\rho) + \frac{B^I}{\mathcal{B}}\sinh(\mathcal{B}\rho)\,, \tag{4.17}$$

which allows us to write eq. (4.14) in the more manageable form

$$v_{KL}\left(B^K B^L - \mathcal{B}^2 A^K A^L - p^K p^L\right) = 0\,. \tag{4.18}$$

In the same way we can write the equations of motion,

$$(\partial_I v_{KL})\left(B^K B^L - \mathcal{B}^2 A^K A^L - p^K p^L\right) = 0\,, \tag{4.19}$$

which can be seen as an extremization condition for eq. (4.18).

In order to make further contact with the supergravity fields, we mimic the definition of the scalar fields in eq. (2.31) by defining

$$\varphi^I \equiv \frac{h^I}{h^0} = \frac{K^I}{K^0}\,, \qquad \text{so that} \qquad \varphi^0 \equiv 1\,. \tag{4.20}$$

As before, we can then fix $A^I$ in terms of the asymptotic values of the scalar fields $\varphi^I_\infty$ by

$$A^I = \mathsf{V}(\varphi_\infty)^{-1/3}\varphi^I_\infty\,. \tag{4.21}$$

Following ref. [3], we can calculate the string tension to be

$$\mathcal{T}_{(1)} = \mathcal{B} + \tfrac{3}{4}B^I_-\tilde{A}_I\,, \qquad \text{where} \qquad B^I_- \equiv B^I - \mathcal{B}A^I \tag{4.22}$$

and we defined $\tilde{A}_I = \lim_{\rho\to 0}\tilde{K}_I$, which by eq. (4.21) satisfy $A^I\tilde{A}_I = 1$. The values of physical quantities on the (outer) horizon are given by the shifted components analogous to the ones defined in eq. (2.59)

$$B^I_+ = B^I + \mathcal{B}A^I\,, \tag{4.23}$$

which allows us to express the string's tension as

$$\mathcal{T}_{(1)} = \tfrac{3}{4}B^I_+\tilde{A}_I - \tfrac{1}{2}\mathcal{B}\,. \tag{4.24}$$

The temperature of the black string is easily calculated to be

$$T_+ = \frac{4\pi}{\sqrt{2\mathcal{B}}}\,\mathsf{V}^{-1/2}(B_+)\,, \tag{4.25}$$

and ref. [3]'s entropy density $\mathcal{S}_+$ is[19]

$$\mathcal{S}_+ = \mathsf{V}^{2/3}(B_+)\,, \qquad \text{whence} \qquad \sqrt{2\mathcal{B}} = 4\pi T_+ \mathcal{S}_+^{3/4}\,, \tag{4.26}$$

if full concordance with the general results obtained in ref. [3].

The metric (4.4), by an extension of the argument that we gave earlier for black holes, can cover also the interior of the inner horizon, except that the rôles of coordinates $t$ and $y$ become interchanged.[20] We can then calculate the temperature and the entropy density on the inner horizon to be

$$T_- = \frac{4\pi}{\sqrt{2\mathcal{B}}}\mathsf{V}^{-1/2}(B_-)\,, \qquad \mathcal{S}_- = \mathsf{V}^{2/3}(B_-) \qquad \text{and} \qquad \sqrt{2\mathcal{B}} = 4\pi T_- \mathcal{S}_-^{3/4}\,. \tag{4.27}$$

## 4.1 Flow equations for black strings

As in section 2.3, we can derive general flow equations. In this case we can use $h^I = e^U K^I = e^{\hat{U}}\hat{K}^I$, where $\hat{K}^I$ is a function of a new coordinate $\hat{\rho}$ such that

$$\frac{\partial \hat{K}^I}{\partial \hat{\rho}} = B^I\,. \tag{4.28}$$

Using then the completeness and orthogonality relations of real special geometry, we find that the above equation is equivalent to the following system of flow equations

$$\partial_{\hat{\rho}}\hat{U} = -e^{\hat{U}}\mathcal{Z}_{\mathrm{m}}(B)\,, \tag{4.29}$$

$$\partial_{\hat{\rho}}\phi^x = -3\,e^{\hat{U}}\partial^x\mathcal{Z}_{\mathrm{m}}(B)\,, \tag{4.30}$$

where we defined the *fake magnetic (dual) central charge* $\mathcal{Z}_{\mathrm{m}}(B) = h_I B^I$. Observe that the above equations are, *mutatis mutandis*, identical to flow equations (2.70, 2.71) for the black holes. This means that as long as we are considering the same kind of ansatz for the seed functions, which is the case, we will find that the above flow equations will lead to a solution of the FGK equations of motion as long as

$$4\mathcal{B}^2\hat{\rho}\,e^{\hat{U}}\partial^x\mathcal{Z}_{\mathrm{m}}(\phi,B) = e^{2\hat{U}}\left[f^{-2}\partial^x V_{\mathrm{st}}(\phi,B) - \partial^x V_{\mathrm{st}}(\phi,p)\right]\,, \tag{4.31}$$

and the non-extremality parameter can be obtained from

$$V_{\mathrm{st}}(\phi_\infty,p) - V_{\mathrm{st}}(\phi_\infty,B) = \mathcal{B}^2 e^{-2U_\infty} = \mathcal{B}^2\,. \tag{4.32}$$

## 4.2 Non-extremal black strings with constant scalars

As in section 3.1, we can consider the non-extremal analog of the doubly extremal string solution, by which we mean a black string-like solution with constant physical scalars. Using the ansatz (4.17) and the shorthand notation of eq. (4.23), we see immediately that

$$\frac{A^I}{A^0} = \frac{B^I}{B^0} = \frac{B^I_\pm}{B^0_\pm}\,. \tag{4.33}$$

[19] $\mathcal{S}_\pm \equiv \mathrm{Area}_\pm/4$, where Area$_\pm$ is the area of the 2-sphere in the near-outer (respectively near-inner)-horizon geometry.

[20] It is perhaps useful to introduce the coordinate $r$ by $r\cdot(1-e^{-2\mathcal{B}\rho}) = 2\mathcal{B}$, which takes the FGK metric in eq. (4.4) to the standard form with a blackening factor.

The general form of the $K$'s then becomes

$$K^I = A^I \,\mathcal{K}\,, \qquad \mathcal{K} = \cosh(\mathcal{B}\rho) + \frac{\mathcal{E}}{\mathcal{B}}\sinh(\mathcal{B}\rho)\,, \tag{4.34}$$

with a constant of proportionality $\mathcal{E} = \frac{B^0}{A^0}$, which must be positive in order for the metric to be well-defined:

$$e^{-U} = [\mathsf{V}(A)]^{1/3}\,\mathcal{K} = \mathcal{K}\,, \tag{4.35}$$

where we used asymptotic flatness: $\mathsf{V}(A) = 1$. In this general case the tension, eq. (4.24), and the entropy densities, eqs. (4.26) and (4.27), can be calculated to be

$$\mathcal{T}_{(1)} = \tfrac{1}{4}(3\mathcal{E} + \mathcal{B})\,, \qquad \mathcal{S}_{\pm} = (\mathcal{E} \pm \mathcal{B})^2\,. \tag{4.36}$$

As always, the precise relation between various constants appearing in the solution, notably $\mathcal{E}$ and $\mathcal{B}$, has to be fixed by the equations of motion (4.19) and the Hamiltonian constraint (4.19). For the case at hand they can be recast in the form

$$\mathcal{E}^2 + V_{\rm st}(A,p) = \mathcal{B}^2\,, \tag{4.37}$$

$$[\partial_I V_{\rm st}](A,p) = 0\,. \tag{4.38}$$

The evident similarity to eqs. (3.7) and (3.8) was to be expected.

## 4.3 Extremal strings

In this case we are interested in solutions for which $\mathcal{B} = 0$. By defining

$$V_{\rm st}(K,B) = -a_{IJ}B^I B^J \qquad \text{and also} \qquad V_{\rm st}(K,p) = -a_{IJ}p^I p^J\,, \tag{4.39}$$

where the $K$-dependence resides in $a_{IJ}$ through eq. (4.12), we can see that the equations of motion and the Hamiltonian constraint can be written as

$$\partial_I\big(V_{\rm st}(K,B) - V_{\rm st}(K,p)\big) = 0 \qquad \text{and} \qquad V_{\rm st}(K,B) = V_{\rm st}(K,p)\,, \tag{4.40}$$

the former being implied by the latter.

The above results are nothing new as they also follow from the flow equations, but it is interesting to evaluate them on the horizon[21]:

$$[\partial_I V_{\rm st}]\,(B,p) = 0 \qquad \text{and} \qquad v_{KL}(B)\,p^K p^L = -3\,. \tag{4.41}$$

### 4.3.1 Extremal strings of the $STU$ model

The 5-dimensional $STU$ model can be obtained as a consistent truncation of a 6-torus compactification of M-theory, meaning that any 5-dimensional solution can always be lifted to M-theory. As is well-known, the supersymmetric black holes derived in sec. 3.2 correspond to the intersection of three M2-branes, which after a chain of dualities leads to *e.g.* a D5-D1-F1 intersection that is used to calculate the microscopic entropy [42]. This identification can also be used to explain the microscopic origin of the near-extremal black holes [43]. The uplift of the supersymmetric strings is readily

[21] These equations are analogous to eqs. (2.44) and (2.42).

identified with an intersection of three M5-branes and the general string solutions can be seen to be deformations of these intersections.

The relevant polynomial reads

$$\mathsf{V}(K) = K^0 K^1 K^2 \,. \tag{4.42}$$

which, when comparing it to the $\mathrm{W}(H)$ in sec. 3.2 and remembering that the Hesse metric is the second derivative of the logarithm of V, means that the problem is completely analogous to the problem treated in section 3.2. It is therefore no great surprise to see that the Hamiltonian constraint and the equations of motion are given respectively by

$$3\mathcal{B}^2 = \sum_I \frac{\ddot{K}^I}{K^I} \,, \tag{4.43}$$

$$0 = K^I \ddot{K}^I - (\dot{K}^I)^2 + (p^I)^2 \quad \text{(no summation)}. \tag{4.44}$$

The general comments made in sec. 3.2 about the separability of the equations and their solutions apply also in this case.

The extremal solutions, taking into account the branches and the signs, are given by harmonic functions

$$K^I = s^I \,\mathrm{sgn}(p^I) \left(|A^I| + |p^I|\rho\right) , \quad (s^I)^2 = 1 \,, \tag{4.45}$$

where the various signs have to satisfy

$$1 = s^0 s^1 s^2 \,\mathrm{sgn}(p^0)\,\mathrm{sgn}(p^1)\,\mathrm{sgn}(p^2) \qquad \text{and} \qquad s^0 s^x = \sigma^x \,\mathrm{sgn}(p^0)\,\mathrm{sgn}(p^x)\,, \tag{4.46}$$

which are completely analogous to eqs. (3.25) and (3.29). The supersymmetric extremal solutions, *i.e.* the ones that extremize the string central charge $\mathcal{Z}_{\mathrm{m}}(\phi, p)$, have signs that satisfy

$$\mathrm{sgn}(p^0) = \sigma^x \,\mathrm{sgn}(p^x) \ . \tag{4.47}$$

Table 3.2.1 can also be applied to this case, and gives the possible sign choices for the $(+,-)$ branch, the second and seventh row being supersymmetric.

The entropy density for the extremal strings is

$$\mathcal{S} = \left|p^0 p^1 p^2\right|^{2/3} . \tag{4.48}$$

The string's tension can be calculated from eq. (4.24) and reads

$$\mathcal{T}_{(1)} = \frac{1}{4|\varphi^1_\infty \varphi^2_\infty|^{2/3}} \left(|\varphi^1_\infty \varphi^2_\infty p^0| + |\varphi^2_\infty p^1| + |\varphi^1_\infty p^2|\right) , \tag{4.49}$$

when expressed in terms of the asymptotic values of the scalar fields $\varphi^x$.

### 4.3.2 Extremal strings in the heterotic $STU$ model

In this section we consider the extremal string solutions to the $STU$ model with a correction, leaving the non-extremal ones for future work as they are much more involved.

The model that we want to consider can be obtained by compatifying heterotic string theory on $K3 \times S^1$ and the fundamental polynomial is given by [44]

$$\mathcal{V}(h^\cdot) = \begin{cases} h^0 h^1 h^2 + \frac{\aleph^2}{3}\left(h^0\right)^3 & \text{for} \quad h^0 < h^1 \,, \\ h^0 h^1 h^2 + \frac{\aleph^2}{3}\left(h^1\right)^3 & \text{for} \quad h^0 > h^1 \,, \end{cases} \tag{4.50}$$

where $\aleph = 1$ has been introduced in order to be able to discuss the $STU$ limit $\aleph \to 0$. The line $h^0 = h^1$ corresponds to the selfdual radius of the circle compactification, where extra massless modes arise; $h^2$ corresponds to the 5-dimensional dilaton [44]. We shall restrict ourselves to the case when $h^I > 0$, hence also $\varphi^I > 0$, and we shall furthermore restrict ourselves to the wedge of moduli space where $h^1 > h^0$, or alternatively $\varphi^1 > 1$, in order not to have to deal with solutions that interpolate between the two wedges.[22]

Let us in passing mention that the BPS black holes based on this model were obtained by Gaida in ref. [46], who showed that there is a quantum constraint on the electric charges. This restriction arises as follows: by eq. (2.2) we see that

$$3h_0 = h^1h^2 + \aleph^2(h^0)^2\,, \qquad 3h_1 = h^0h^2\,, \qquad 3h_2 = h^0h^1\,, \tag{4.51}$$

which can be inverted over the complex numbers to give

$$h^1 = \frac{3h_2}{h^0}\,, \qquad h^2 = \frac{3h_1}{h^0}\,, \qquad \tfrac{2}{3}\aleph^2(h^0)^2 = h_0 \pm \sqrt{h_0^2 - 4\aleph^2 h_1 h_2}\,. \tag{4.52}$$

Since the $h$'s must be real,

$$(h_0)^2 \geq 4\aleph^2 h_1 h_2 \qquad \text{or dually:} \qquad \big(h^1h^2 - \aleph^2(h^0)^2\big)^2 \geq 0\,, \tag{4.53}$$

to which we shall refer to as Gaida's bound and which is a restriction originating from the well-definedness of the model in real special geometry. As the restriction must also hold on the horizon, the attractor mechanism implies Gaida's constraint $q_0^2 \geq 4\aleph^2 q_1 q_2$ [46].

The supersymmetric solutions can be found easily by extremizing the string central charge $\mathcal{Z}_{\rm m}(p)$. To that end one would in principle need a parameterization of the $h$'s in terms of the physical scalars $\phi^x$, but it is advantageous to use the $\varphi^x$ as physical scalars as then the attractor equation

$$0 = \frac{\partial \mathcal{Z}_{\rm m}(p)}{\partial \varphi^x}\bigg|_{\varphi^x_{\rm h}}\,, \tag{4.54}$$

becomes readily solvable by seeing that $h^x = h^0\varphi^x$ and $(h^0)^{-3} = \mathsf{V}(\varphi^I)$. Perhaps surprisingly, this equation has two solutions, namely:

a) The first solution is given by

$$\varphi^1_{\rm h} = \frac{B^1}{B^0} = \frac{p^1}{p^0}\,, \qquad \varphi^2_{\rm h} = \frac{B^2}{B^0} = \frac{p^2}{p^0}\,, \tag{4.55}$$

which is a solution for the chosen wedge if $\mathrm{sgn}(p^1p^0) = 1$ and $|p^1| > |p^0|$.

The above fixes the scalars on the horizon, but does not give us the $B$-coefficients; for that we need to solve eq. (4.32), which for generic charges gives $B^0 = s^0 p^0$, where $s^0 = \pm 1$ as is customary in this article. Given this identification we can then calculate the entropy density, whose positivity implies that $s^0 = \mathrm{sgn}(p^0)$:

$$\mathcal{S}^{3/2}(p) = |p^0p^1p^2| + \tfrac{\aleph^2}{3}|p^0|^3\,. \tag{4.56}$$

The sign of $B^0$ together with the sign restrictions on the scalars determine the $B$'s to be $B^I = |p^I|$.

[22] The interesting case of having a solution that switches from one wedge to another will not be considered here. See *e.g.* [45] for supersymmetric black hole and string solutions that do switch wedges.

Due to the fact that the magnetic charges must all have the same sign, the equations of motion (4.19) are satisfied for all values of $A^I$, so we can take $K^I = |A^I|+|p^I|\rho$ to ensure the regularity of the resulting solution and impose the normalization condition $\mathsf{V}(|A|) = 1$ in order to obtain an asymptotic Minkowski metric. The final constraint comes from the fact that $\varphi^1(\rho) > 1$: it is easily seen that this is satisfied if and only if $|A^1| > |A^0|$.

A string in this class that saturates Gaida's bound, satisfies $p^1p^2 = \aleph^2|p^0|^2$, and the resulting entropy density is

$$\mathcal{S}|_{\text{Gaida}} = \left(\tfrac{4}{3}\right)^{2/3}|p^0|^2\,. \tag{4.57}$$

b) The second solution has no *classical*, *i.e.* $\aleph^2 \to 0$, limit and exists if and only if $\operatorname{sgn}(p^1p^2) = 1$. It reads

$$\varphi^1_{\text{h}} = \frac{B^1}{B^0} = \sqrt{\frac{|p^1|}{|p^2|}\aleph^2}\,, \qquad \varphi^2_{\text{h}} = \frac{B^2}{B^0} = \sqrt{\frac{|p^2|}{|p^1|}\aleph^2}\,, \tag{4.58}$$

which lies in the desired wedge if $|p^1| > |p^2|$. Observe that this solution saturates Gaida's bound on the horizon.

The Hamiltonian constraint on the horizon fixes

$$B^0 = \tfrac{1}{2}s^0\operatorname{sgn}(p^0)\left(|p^0| + \operatorname{sgn}(p^0p^1)\sqrt{|p^1p^2|}\right). \tag{4.59}$$

The equations of motion show that they are satisfied iff $p^1p^2 = \aleph^2|p^0|^2$, which immediately reduces this case to the Gaida solution of case a).

The extremal non-BPS solutions to this case are not as easy to find as in the $STU$ model,[23] and one has to resort to a different approach: first we solve the equations (4.41) in order to find the relation between the $B$'s and the $p$'s and then solve the full equations of motion. Clearly, solving the first of eqs. (4.41) for the $B$'s is challenging, but seeing that it is quadratic in $p$'s, we first solve it to obtain $p = p(B)$ and then try to invert this relation.

Now there are four cases that solve the first of the eqs. (4.41), one of which corresponds to the BPS solution above and three correspond to extremal non-BPS solutions:

i) The first case is given by

$$\varphi^1_{\text{h}} = \frac{B^1}{B^0} = -\frac{p^1}{p^0} - \frac{2\aleph^2p^0}{3p^2}\,, \qquad \varphi^2_{\text{h}} = \frac{B^2}{B^0} = \frac{p^2}{p^0}\,. \tag{4.60}$$

For this solution to be valid in the chosen wedge we must have that $\operatorname{sgn}(p^0) = \operatorname{sgn}(p^2) = -\operatorname{sgn}(p^1)$ and the magnetic charges must be such that

$$|p^1| > |p^0| + \frac{2|p^0|^2}{3|p^2|}\,. \tag{4.61}$$

The normalization condition then gives $(p^0)^2 = (B^0)^2$ and we can calculate

$$\mathcal{S}^{3/2} = B^0\left(|p^1p^2| - \tfrac{\aleph^2}{3}|p^0|^2\right). \tag{4.62}$$

[23] The system has a discrete symmetry with respect to the interchange of indices 1 and 2. The function $\mathcal{V}$ has the more important scaling symmetry $h^1 \to e^{\lambda}h^1$ and $h^2 \to e^{-\lambda}h^2$, but it does not leave the equations of motion in the H-formalism invariant.

As one can see, due to the restriction on the charges, we have that the term between the parentheses is positive, so we need to choose $B^0 = |p^0|$, implying

$$\mathcal{S}^{3/2} = |p^0 p^1 p^2| - \tfrac{\aleph^2}{3}|p^0|^3 \,. \tag{4.63}$$

Surprisingly, the Hamiltonian constraint does not impose any condition on $A^1$ but imposes the condition $p^2 A^0 = p^0 A^2$. We solve this condition by introducing a positive number $\beta$ and writing

$$A^0 = \beta |p^0| \,, \qquad A^2 = \beta |p^2| \,, \qquad \text{thus} \qquad \varphi^2(\rho) = \frac{|p^2|}{|p^0|} \,. \tag{4.64}$$

Even more surprisingly, the equations of motion are identically satisfied by the above relations between the coefficients $A$ and $B$.

With the above information we can then calculate the metrical function

$$e^{-3U} = |p^0|\,(\beta+\rho)^2 \left[ |p^2| A^1 + \tfrac{\aleph^2}{3}\beta |p^0|^2 + \left( |p^1 p^2| - \tfrac{\aleph^2}{3}|p^0|^2 \right)\rho \right] . \tag{4.65}$$

Its regularity becomes more manifest when we impose the asymptotic Minkowskianity condition

$$|p^2| A^1 + \tfrac{\aleph^2}{3}\beta |p^0|^2 = \frac{1}{\beta^2 |p^0|} \,, \tag{4.66}$$

which allows us to express the metrical factor as

$$e^{-3U} = \left(1 + \beta^{-1}\rho\right)^2 \left[ 1 + \beta^2 \left( |p^0 p^1 p^2| - \tfrac{\aleph^2}{3}|p^0|^3 \right)\rho \right] . \tag{4.67}$$

The tension of this string is easily calculated to give

$$4\mathcal{T}_{(1)} = 2\beta^{-1} + \beta^2 \left( |p^0 p^1 p^2| - \tfrac{\aleph^2}{3}|p^0|^3 \right) = 2\beta^{-1} + \beta^2 \mathcal{S}^{3/2} \,, \tag{4.68}$$

which is always positive, due to the restrictions imposed on the magnetic charges; the minimal attainable tension occurs when $\beta = \mathcal{S}^{-1/2}$, from which we have that $\mathcal{T}_{(1)} \geq \frac{3}{4}\mathcal{S}^{1/2}$.

We have seen that $\varphi^2(\rho)$ is just a constant and that it always satisfies $\varphi^2 > 0$. The situation with $\varphi^1$ is slightly more complicated as it must satisfy $\varphi^1(\rho) > 1$. Writing out the constraint we see that

$$\beta |p^0|^2 \left( |p^2| + \tfrac{\aleph^2}{3}|p^0| \right) - \beta^{-2} \leq \rho |p^0 p^2| \left( |p^1| - |p^0| - \frac{2|p^0|^2}{3|p^2|} \right) . \tag{4.69}$$

Since the term in brackets on the right-hand side is positive due to the condition for the scalar on the horizon to be in the correct wedge and since $\rho \in [0,\infty)$, we see that the left-hand side must in fact be smaller than zero, or

$$\beta^3 \left( |p^0|^2 |p^2| + \tfrac{\aleph^2}{3}|p^0|^3 \right) < 1 \,. \tag{4.70}$$

ii) This case is readily obtained from case i) by using the obvious symmetry of the equations of motion and the Hamiltonian constraint under the interchange of the indices 1 and 2. What is not invariant under this change is the choice of wedge, which means that the restrictions we need to impose will be different from the ones imposed in case i). The solution to eq. (4.41) is

$$\varphi^1_{\mathrm{h}} = \frac{B^1}{B^0} = \frac{p^1}{p^0} \,, \qquad \varphi^2_{\mathrm{h}} = \frac{B^2}{B^0} = -\frac{p^2}{p^0} - \frac{2\aleph^2 p^0}{3p^1} \,. \tag{4.71}$$

The choice of wedge then implies that $\text{sgn}(p^1) = \text{sgn}(p^0)$ and that $|p^1| > |p^0|$; the fact that $\varphi^2 > 0$ then implies that $\text{sgn}(p^2) = -\text{sgn}(p^0)$ and

$$|p^1| > |p^0| \qquad \text{and} \qquad |p^2| > \frac{2\aleph^2|p^0|^2}{3|p^1|}\,. \tag{4.72}$$

The normalization condition in eq. (4.41) gives $B^0 = s^0 p^0$ and as before the sign $s^0$ is fixed by the entropy density to be $s^0 = \text{sgn}(p^0)$; the resulting entropy density is identical to the one in eq. (4.63) and is positive owing to the restrictions (4.72).

Similarly to what happened before, the Hamiltonian constraint and the equations of motion impose no condition on $A^2$, but impose the condition $A^1|p^0| = A^0|p^1|$. We solve it by $A^0 = \gamma|p^0|$ and $A^1 = \gamma|p^1|$, which immediately implies that $\varphi^1(\rho) = |p^1|/|p^0| > 1$, so there is no possibility of the solution leaving the chosen wedge of moduli space.

Regularity of the warp factor in ensured by the asymptotic Minkowskianity condition, which not only fixes

$$\gamma^2|p^0p^1|A^2 = 1 - \tfrac{\aleph^2}{3}\gamma^3|p^0|^3 \tag{4.73}$$

and brings the metrical factor to the form in eq. (4.67), but also means that the tension of the string in this case is the one in eq. (4.68). The final ingredient then is the condition that $\varphi^2(\rho)$ be strictly positive. This condition is easily calculated and gives

$$\tfrac{\aleph^2}{3}\gamma^3|p^0|^3 < 1\,. \tag{4.74}$$

iii) The third non-supersymmetric solution to eqs. (4.41) can be found by imposing $p^1B^2 = p^2B^1$, from which one finds that $\varphi^1_{\text{h}}$ must satisfy the fourth-order equation

$$2p_0p_1^2(\varphi^1_{\text{h}})^4 + 7p_1^2p_2(\varphi^1_{\text{h}})^3 - 3p_0p_1p_2(\varphi^1_{\text{h}})^2 - 3p_1p_2^2\varphi^1_{\text{h}} - 3p_0p_2^2 = 0\,. \tag{4.75}$$

The solutions are however too intricate to be of any real use and this case will, therefore, not be treated.

# 5 Conclusions

We have extended the H-FGK formalism of [15] to black strings and applied it to find examples of black-hole and black-string solutions in specific models, as well as re-derive a non-extremal solution with constant scalars discussed earlier by [13], which is a solution to any model of $N = 2$ supergravity in five dimensions, coupled to vector multiplets. Since strings couple magnetically rather than electrically to the gauge fields, the rôles of primary (untilded) and dual (tilded) $H$-variables are interchanged in comparison with the case of black holes (for distinction we denoted the primary variables $K$ when discussing strings). In the $STU$ model, however, the resulting equations for black strings are the same as for black holes.

The model-independent relationship between a set of parameters appearing in the H-formalism and the asymptotic values of the scalars is a significant simplification with respect to the original FGK formulation (in physical variables), where the parameters need to be determined in each case from complicated equations. For extremal solutions, the other set of parameters, which can be called fake charges, is given by the condition that the black hole potential be stationary, this way completing a simple procedure for constructing extremal (supersymmetric and non-supersymmetric) black hole solutions.

The derivation of first-order flow equations for non-supersymmetric extremal black holes and black strings presented here allows the relation between the fake and actual charges to be non-linear, which is indeed the case in the specific example of the model from a Jordan sequence. For non-extremal solutions, the hyperbolic ansatz makes it possible to bring the flow equations to the same form as the extremal flow. On the other hand, one could argue that once the harmonicity or hyperbolicity assumptions have been adopted, the analysis of flow equations as such becomes perhaps superfluous, since the radial profile of the scalars is already established by the respective ansätze.

We have demonstrated that for the $STU$ model in five dimensions the hyperbolic (or exponential) ansatz in the non-extremal case and the harmonic ansatz in the extremal case correspond to the most general solutions of the equations of motion. We expect these ansätze for the variables $H$ and $K$ to be valid in all five- and four-dimensional models for all static solutions with transverse spherical symmetry. For non-static black holes a less restrictive ansatz is required, as the four-dimensional stabilization equations for general extremal black holes suggest [32].

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